New basis function approach to the 't Hooft-Bergknoff-Eller equations

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We analytically and numerically investigate the 't Hooft-Bergknoff-Eller equations, the lowest order mesonic light-front Tamm-Dancoff equations for U(N_C) and SU(N_C) gauge theories. We find the wave function can be well approximated by new basis functions and obtain an analytic formula for the mass of the lightest bound state. Its value is consistent with the precedent results. [S0556-2821(99)01020-6]

I. INTRODUCTION

Light-front (LF) quantization is believed to be an effective method for studying many-body relativistic field theories [1,2]. The physical vacuum is equivalent to the bare vacuum in the LF coordinate, since all constituents must have non-negative longitudinal momenta defined by $k^+ = (k^0 + k^3)/\sqrt{2}$. This simple structure of the true vacuum enables us to avoid the serious problems which appeared in the Tamm-Dancoff (TD) approximation [3] in the equal time frame. Therefore, the TD approximation [4] is commonly used in the context of LF quantization.

The techniques have been developed [5–7] for solving LFTD equations in several models such as the massive Schwinger model [8], which is the extension of the simplest $(1+1)$-dimensional QED (QED$_2$) [9]. Bergknoff [10] first applied the LFTD approximation to the massive Schwinger model. He obtained the so-called Bergknoff equation, which is the light front Einstein-Schrödinger equation truncated to one fermion-antifermion pair. He obtained excellent results for the lowest energy meson under 't Hooft's ansatz. He also discovered that it is necessary to include two fermion-antifermion pairs in order to study the excited states.

Ma and Hiller [5] studied the lowest Bergknoff equation numerically. They developed their numerical method for solving the Bergknoff equation using an idea based upon 't Hooft's ansatz.

Mo and Perry [6] suggested that even in the first excited state, most of the wave function consists of four-fermion sectors when the fermion mass reaches zero. Therefore, four or more fermion sectors must be included so as to describe wave functions beyond the ground state bosons.

In Ref. [6], Mo and Perry presented an effective way to treat the ground state and the excited state in the massive Schwinger model. They concluded that to study the massive Schwinger model, the Jacobi polynomials are suitable as basis functions. Harada *et al.* [7] studied the massive Schwinger model with SU(2) flavor symmetry, including up to four fermion sectors. They used simpler basis functions, which are essentially equivalent to the Jacobi polynomials. Sugihara, Matsuzaki, and Yahiro [11] numerically analyzed 2-dimensional SU(N_C) quantum chromodynamics (QCD)

[12], including four-fermion sectors, by means of the basis functions of Harada *et al.*

Although excellent papers exist concerning massless and massive Schwinger models and two-dimensional QCD [16–19], including excited states, it is worth analyzing the "'t Hooft-Bergknoff-Eller'' equation [13], the extension of the 't Hooft-Bergknoff equation, in order to include both SU(N_C) and U(N_C) gauge theories. This is because there is a mathematical interest in the basis function method. There is no mathematical evidence that the conventional basis function expansion describes the wave function well; instead, the contrary is rather to be supposed, as there is evidence that the conventional method breaks down if we try to improve the approximation. We would therefore like to improve the basis functions so as to avoid such difficulties.

Further, there is a so-called "2% discrepancy" problem, briefly summarized as follows: We expand dimensionless meson mass squared M^2 in terms of dimensionless quark mass m:

$$M^2 = 1 + b_1 m + b_2 m^2 + \cdots, \tag{1.1}$$

where

$$M^2 = \frac{\bar{M}^2}{\mu(N_C)} \quad \text{and} \quad m^2 = \frac{\bar{m}^2}{\mu(N_C)}. \tag{1.2}$$

Here, $\bar{M}$ is a mass of bound state, $\bar{m}$ denotes a bare mass of quark, and $\mu(N_C)$ stands for $\mu(N_C) = (N_C^2 - 1 + \alpha)g^2/2\pi N_C$. That is, we measure all masses in the unit of $\mu(N_C)$. Banks *et al.* [14] obtained first order coefficient b_1 analytically, using the bosonization method:

$$b_1 = 2e^{\gamma_E} = 3.56214\cdots, \tag{1.3}$$

where $\gamma_E = 0.57721\cdots$ is Euller's constant. On the other hand, Bergknoff [10] found the value

$$b_1 = \frac{2\pi}{\sqrt{3}} = 3.62759\cdots, \tag{1.4}$$

which differs from Eq. (1.3) by 2%. In Ref. [15], the authors suggested that the coefficient $b_1 = 2\pi/\sqrt{3}$ was a variational invariant and that this discrepancy was ascribable to the contributions from the higher Fock sectors. Before proceeding on to consider the higher Fock sectors, we have to examine

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whether this discrepancy can be explained or not in terms of the lowest light-front Tamm-Dancoff equation using all possible basis functions.

The 't Hooft-Bergknoff-Eller equation for two-dimensional gauge theory is given in the form

$$M^2\Phi(x) = \int_{-1/2}^{1/2} dy\, H(x,y)\Phi(y)$$

$$\equiv \frac{4(m^2-1)}{1-4x^2}\Phi(x) - \wp \int_{-1/2}^{1/2} dy\, \frac{\Phi(y)}{(y-x)^2}$$

$$+ \alpha \int_{-1/2}^{1/2} dy\, \Phi(y), \quad -\frac{1}{2} \leq x \leq \frac{1}{2}, \quad (1.5)$$

where $\wp$ denotes the finite part integral, $\alpha=1$ for U(N_C), and $\alpha=0$ for SU(N_C). In Eq. (1.5), we shifted the variable x total amount of $-\frac{1}{2}$ compared with the variable in Refs. [10,12,13], in order to show the symmetry of the wavefunction transparently.

Mo and Perry concluded in Ref. [6] that the Jacobi polynomials, $(1-4x^2)^\beta P_n^{\beta,\beta}(2x)$ in our notation, are the most suitable basis functions. This conclusion seems quite natural because the system of the Jacobi polynomials $P_n^{\beta,\beta}(2x)$ is an orthogonal complete set on the interval $-\frac{1}{2} \leq x \leq \frac{1}{2}$ with respect to the weight function $(1-4x^2)^\beta$. Harada and collaborators [7] suggested using the simpler basis functions, $(1-4x^2)^{\beta+j}$ and $x(1-4x^2)^{\beta+j}$ in our notation, which are equivalent to the abovementioned ones. According to Harada *et al.*, one can expect that the wave function could be expanded as follows:

$$\Phi(x) = \begin{cases} \lim_{N\to\infty} \sum_{j=0}^N a_j(1-4x^2)^{\beta+j}, \\ \\ \lim_{N\to\infty} \sum_{j=0}^N b_j x(1-4x^2)^{\beta+j}. \end{cases} \quad (1.6)$$

Here, we have used the fact, as is shown in the Appendix, that the Eller equation does not mix even and odd functions with each other.

The exponent β and the quark mass m are related to each other by the equation [12,10]

$$(m^2-1) + \beta\pi \cot\beta\pi = 0. \quad (1.7)$$

The authors of Refs. [5,10,7,11] adopted the positive smallest solution $\beta_0(m)$ of Eq. (1.7) as β in Eq. (1.6). That is, for small m,

$$\beta_0(m) = \sqrt{\frac{3m}{\pi}}\left(1 - \frac{m^2}{10}\right) + O(m^5) \equiv \beta_0^1 m + \beta_0^3 m^3 + \cdots. \quad (1.8)$$

In the following sections, we try to determine the coefficients a_n's, according to the predecessors.

II. CONVENTIONAL BASIS FUNCTION METHOD

In this section, we restrict ourselves to the case $\alpha=0$ where the gauge group is SU(N_C). Mo and Perry, and Harada and his collaborators, presented an effective way to determine the coefficients. We will briefly reproduce their procedures. By the use of the expansion in Eq. (1.6) truncated to given finite number N for the wave function Φ, we multiply both sides of Eq. (1.5) by $(1-4x^2)^{\beta+i}$ and integrate them over x, then we obtain

$$M^2\hat{N}\vec{a} = \hat{H}\vec{a}, \quad \vec{a} = {}^t[a_0, a_1, \ldots, a_{n-1}]. \quad (2.1)$$

Here $\hat{N}$ and $\hat{H}$ are $n\times n$ matrices and are given by

$$\hat{N}_{ij} = \int_{-1/2}^{1/2} dx (1-4x^2)^{\beta+i}(1-4x^2)^{\beta+j}$$

$$= \frac{\pi^{1/2}\Gamma(2\beta+i+j+1)}{2\Gamma(2\beta+i+j+3/2)} \quad (2.2)$$

and

$$\hat{H}_{ij} = 4(m^2-1)\int_{-1/2}^{1/2} dx (1-4x^2)^{2\beta+i+j-1}$$

$$- \wp \int_{-1/2}^{1/2} dx\, dy\, \frac{(1-4x^2)^{\beta+i}(1-4y^2)^{\beta+j}}{(y-x)^2}$$

$$= 2\pi^{1/2}(m^2-1)\frac{\Gamma(2\beta+i+j)}{\Gamma(2\beta+i+j+1/2)}$$

$$+ \frac{2^{4\beta+2i+2j-3}(\beta+i)(\beta+j)}{2\beta+i+j}$$

$$\times B(\beta+i,\beta+i)B(\beta+j,\beta+j), \quad (2.3)$$

see the Appendix of Ref. [7]. So-called ''norm'' matrix $\hat{N}$ appeared in the above equation because the basis functions we have used are not orthonormalized. In order to have eigenvalues of the generalized eigenvalue equation given in Eq. (2.1), we have to solve the eigenvalue problem for norm $\hat{N}$ first, i.e.,

$$\hat{N}\vec{v}_i = \lambda_i \vec{v}_i. \quad (2.4)$$

Next, we introduce a transformation matrix $\hat{W}$ by

$$\hat{W} = \left[\frac{\vec{v}_1}{||\vec{v}_1||\sqrt{\lambda_1}} \cdots \frac{\vec{v}_n}{||\vec{v}_n||\sqrt{\lambda_n}}\right]. \quad (2.5)$$

Then, we can transform Eq. (2.1) into a usual eigenvalue problem of the form

$$M^2\vec{b} = {}^t\hat{W}\hat{H}\hat{W}\vec{b}, \quad \vec{a} = \hat{W}\vec{b}. \quad (2.6)$$

We can solve Eq. (2.6), numerically. For $N=3$ and $m=0.01$, we find, for the ground state boson

$$\beta=0.00552328, \quad M^2=0.0366342,$$

$$a_0=1, \quad a_1=0.00203562,$$

$$a_2=-0.000579369, \quad a_3=0.000165813. \quad (2.7)$$

The values of the left-hand side (LHS) and the right-hand side (RHS) of Eq. (1.5) are shown in Fig. 1. The coincidence of the LHS and the RHS is high for small values of x. For $x \simeq \pm 1/2$, the behavior of the LHS and the RHS are quite different. There are sharp spikes at the end points. This behavior is not changed much even if we improve the order of approximation.

Note here that in order to solve the generalized eigenvalue problem, the norm matrix should be positive definite. We cannot advance the above procedure beyond $N \simeq 12$, because some of the eigenvalues of the norm matrix $\hat{N}$ become almost zero or negative. We will examine mathematically this approximated wave function in detail in the next section.

III. AN INSPECTION OF THE CONVENTIONAL BASIS FUNCTION METHOD

A. Behavior of wave function around $x=0$

We introduce linear map $\mathcal{L}$ by

$$\mathcal{L}:f \mapsto \mathcal{L}f \text{ such that } (\mathcal{L}f)(x)=\wp \int_{-1/2}^{1/2} dy \, \frac{f(y)}{(y-x)^2}. \quad (3.1)$$
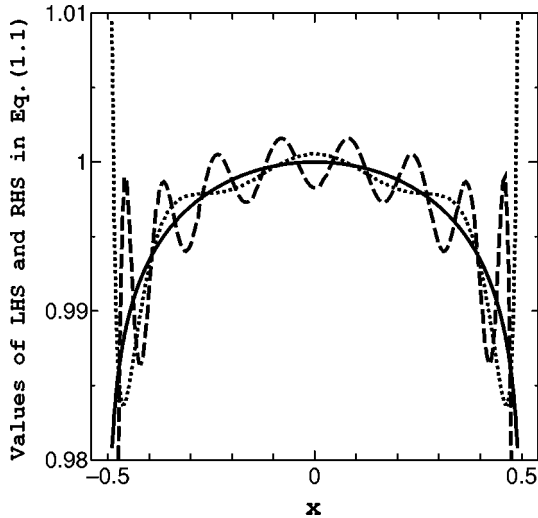

FIG. 1. The comparison of the relative values of both sides of Eq. (1.5) for $\alpha=0$. The wave function in Eq. (1.5) was approximated by Eq. (1.6) with $N=3$ and Eq. (2.7). The solid line represents the LHS and the dotted line stands for the RHS. The RHS with $N=9$, which is indicated by the dashed line, is exhibited for comparison.

After some tedious but not particularly difficult calculations, we find that

$$\mathcal{L}:(1-4x^2)^\beta \mapsto -4\pi^{1/2}\frac{\Gamma(\beta+1)}{\Gamma(\beta+1/2)}F(1,1/2-\beta;1/2;4x^2),$$

$$\mathcal{L}:x(1-4x^2)^\beta \mapsto -8\pi^{1/2}\frac{\Gamma(\beta+1)}{\Gamma(\beta+1/2)}xF(2,1/2-\beta;3/2;4x^2). \quad (3.2)$$

where $F(a,b;c;x)$ is the Gauss' hypergeometric function [20], see the Appendix.

We restrict ourselves to the case where the wave function is an even function, because we are interested in only the ground state meson. Thus, we are led to

$$\lim_{N\to\infty}\sum_{n=0}^N a_n M^2(1-4x^2)^{\beta+n}$$

$$=\lim_{N\to\infty}\sum_{n=0}^N a_n\left[4(m^2-1)(1-4x^2)^{\beta+n-1}\right.$$

$$\left.+4\pi^{1/2}\frac{\Gamma(\beta+n+1)}{\Gamma(\beta+n+1/2)}F(1,1/2-\beta-n;1/2;4x^2)\right]. \quad (3.3)$$

Now, we can examine whether the numerical result obtained so far satisfies Eq. (1.5) or not. We substitute Eq. (2.7) into the above equation, and obtain

$$\text{LHS}(x)=0.0366936-0.00101207x^2-0.00165597x^4$$
$$-0.00466671x^6+O(x^8) \quad (3.4)$$

and

$$\text{RHS}(x)=0.0367133-0.0038283x^2+0.0503141x^4$$
$$-0.210419x^6+O(x^8). \quad (3.5)$$

Thus, the LHS and the RHS coincide, within numerical errors, with each other only up to $O(x^0)$. We calculated coefficients a_n's up to $n=10$, but the coincidence between the LHS and the RHS is not much improved. See Fig. 1.

B. Behavior of wave function near $x=\pm\frac{1}{2}$

Bergknoff suggested that the behavior of wave function Φ near end points $x=\pm\frac{1}{2}$ is important in order to calculate the mass eigenvalue M^2. In fact, Eq. (1.7) is derived by demanding that the most singular part, that is, the coefficients of $(1-4x^2)^{\beta-1}$ on the RHS in Eq. (1.5) must be cancelled, as there is no such term on the LHS. According to Bergknoff's suggestion, we will examine the behavior of wave function near the end points beyond $O((1-4x^2)^{\beta-1})$. We set $4x^2=1-\epsilon$. By the use of the identity for the hypergeometric functions, that is, Eq. (15.3.6) in Ref. [20], which is valid for $a+b-c\neq$ integer, we can expand the RHS of Eq. (3.3) around $\epsilon=0$ and have

$$\text{LHS} = M^2 \sum_{n=0}^{\infty} a_n \epsilon^{\beta+n} \tag{3.6}$$

and

$$\text{RHS} = 4(m^2-1) \sum_{n=0}^{\infty} a_n \epsilon^{\beta+n-1} + 4\pi \sum_{n=0}^{\infty} \sum_{j=0}^{n} a_j(\beta+j)$$

$$\times \cot[\pi(\beta+j)] \frac{(-1/2)_{n-j}}{(n-j)!} \epsilon^{\beta+n-1} - 2\pi^{1/2} \sum_{n=0}^{\infty}$$

$$\times \left[\sum_{j=0}^{\infty} a_j \frac{\Gamma(\beta+j+1)(1/2-\beta-j)_n}{(\beta+j-1)\Gamma(\beta+j+1/2)(2-\beta-j)_n} \right] \epsilon^n. \tag{3.7}$$

Substituting the numerical solution, which is given in Eq. (2.7), into Eqs. (3.6) and (3.7), we have

$$\text{LHS}(\epsilon) = 0.0366342\epsilon^{\beta} + 7.45734\times10^{-5}\epsilon^{\beta+1} - 2.12247$$

$$\times 10^{-5}\epsilon^{\beta+2} + O(\epsilon^{\beta+3}) \tag{3.8}$$

and

$$\text{RHS}(\epsilon) = 0.551656 - 0.525734\epsilon^{\beta} + 2.09072\epsilon - 2.08014\epsilon^{\beta+1}$$

$$+ O(\epsilon^2). \tag{3.9}$$

Only $\epsilon^{\beta-1}$ terms on the LHS and the RHS coincide with each other, because we define β so that the coefficients of $\epsilon^{\beta-1}$ cancel each other on the RHS. Note that the LHS does not contain ϵ^n terms with non-negative integer n, while the RHS does. These tendencies are not changed even if we calculate a_n's for $n=3$, 6, and 10.

If we rewrite the first two terms of Eq. (3.9) as

$$0.551656(1-\epsilon^{\beta}) + 0.025922\epsilon^{\beta}, \tag{3.10}$$

we can see the origin of the spikes at the endpoints in Fig. 1. The spikes arise from the existence of the constant term in the wave function. The wave function $\Phi(x) \equiv 1$ is the exact solution of Eq. (1.5) for $m^2=0$ and $M^2=0$ in SU(N_C). Thus, one may expect that the spikes at the end points are closely related to the existence of the massless bound state in SU(N_C). This is not the case, however, because we can easily see that the constant term, in the wave function, is allowed if and only if $m^2 \equiv 0$. We may conclude that the spikes are nothing but the artifact which arose from the fact that we have used the improper basis functions in Eq. (1.6). In fact, we may remove the spikes if we use the suitable wave function. Refer to the next section.

The basis function given in Eq. (1.6) cannot be a good mathematical approximation of the true wave function. The reason is as follows: If we truncate the series Eq. (1.6) to N, we expect that Eq. (1.5) holds up to $O(\epsilon^{\beta+N-1})$. We have only $N+1$ parameters a_1, a_2, ..., a_n and M^2. On the other hand, we have $2N$ equations up to $O(\epsilon^{\beta+N-1})$. No consistent solution can exist in general in this case.

The main difficulty comes from the fact that $\mathcal{L}$ maps $(1-4x^2)^{\beta}$ not only to the terms $(1-4x^2)^{\beta+j-1}$ with non-negative integer j but also to the terms $(1-4x^2)^j$. One may expect that the above difficulty is avoidable if the additional terms $(1-4x^2)^j$ are introduced in Eq. (1.6). However, we can easily see that coefficients of such terms must cancel. By the use of the identity, that is Eq. (15.3.11) in Ref. [20], we have, for $n \geq 2$,

$$\mathcal{L}:(1-4x^2)^n \mapsto -\frac{1}{2(n-1)} \sum_{k=0}^{n-2} \frac{(1/2-n)_k}{(2-n)_k}(1-4x^2)^k$$

$$-\frac{\pi^{1/2}}{\Gamma(1/2-n)}(-1+4x^2)^{n-1}$$

$$\times \sum_{k=0}^{\infty} \frac{(n)_k(-1/2)_k}{k!(k+n-1)!}(1-4x^2)^k$$

$$\times [\log(1-4x^2) - \psi(k+1)$$

$$+ \psi(-1/2+k)]. \tag{3.11}$$

Here, $\psi(z)$ denotes the digamma function. An analogous formula for $n=1$ holds. See Eq. (15.3.10) in Ref. [20]. If we introduce a term $(1-4x^2)^n$ in Eq. (1.6) with positive integer n, a new singular term such as $\epsilon^{n-1}\log\epsilon$ appears only on the RHS. Note that the exponent n of ϵ is the same as that of the introduced term. Thus all the coefficients of $(1-4x^2)^n$ should be zero.

IV. NEW BASIS FUNCTION

We must notice that there are infinite solutions of Eq. (1.7) in addition to the solution given by Eq. (1.8). In fact, we see that

$$\beta_n(m) = n+1/2 - \frac{1}{(n+1/2)\pi^2} - \frac{2}{3(n+1/2)^3\pi^4}$$

$$+ O\left(\frac{1}{(n+1/2)^5\pi^6}\right) + m^2\left[\frac{1}{(n+1/2)\pi^2}\right.$$

$$\left. + \frac{1}{3(n+1/2)^3\pi^4} + O\left(\frac{1}{(n+1/2)^5\pi^6}\right)\right] + O(m^4)$$

$$\equiv \beta_n^0 + \beta_n^2 m^2 + \cdots, \quad n=1,2,3\cdots. \tag{4.1}$$

From Eqs. (1.8) and (4.1), we are led to

$$0 \ll \beta_n(m) - \beta_0(m) - n < 1/2,$$

$$0 < \beta_n(m) - \beta_k(m) - (n-k) \ll 1, \quad n>k. \tag{4.2}$$

The above relations imply that Eq. (1.6) never incorporates terms such as $(1-4x^2)^{\beta_n+j}$ with positive integer n and non-negative integer j.

We posit that the wavefunction is given by an infinite series

TABLE I. Number of nontrivial equations.

Terms	Range of n	Range of j	Number of nontrivial equations
$(1-4x^2)^{\beta_n-1}$	$0\sim N$		0 (automatically satisfied)
$(1-4x^2)^{\beta_n+j}$	$0\sim N$	$0\sim N-n-1$	$\dfrac{N(N+1)}{2}$
$(1-4x^2)^j$		$0\sim N-1$	N

$$\Phi(x)=\lim_{N\to\infty}\sum_{n=0}^{N}\sum_{j=0}^{N-n}c_n^j(1-4x^2)^{\beta_n(m)+j}. \quad (4.3)$$

For counting the number of free parameters and the number of nontrivial equations, we consider the truncated wave function to given finite N. The truncated wave function includes the term $(1-4x^2)^{\beta_N}$ and all the other lower order terms. We require that Eq. (1.5) should hold up to $O(1-4x^2)^{\beta_N-1}$. For each given value of m, the unknown parameters are M^2 and c_n^j except for $c_0^0\equiv1$. Thus the number of the parameters is $(N+1)(N+2)/2$. On the other hand we have $N(N+3)/2$ nontrivial equations. See Table I.

The number of parameters is larger than that of non-trivial equations by 1. Thus, we can solve the equations for c_n^j in terms of M^2. Another equation of use to us is obtained by multiplying both sides of Eq. (1.5) by $\Phi(x)$ and integrating them over x,

$$M^2\int_{-1/2}^{1/2}dx|\Phi(x)|^2=\int_{-1/2}^{1/2}\int_{-1/2}^{1/2}dx\,dy\,\Phi(x)H(x,y)\Phi(y). \quad (4.4)$$

It should be noted here that Eq. (4.3) is, mathematically, the most general expansion. This means that there is no room to introduce any other additional terms such as $d(1-4x^2)^\gamma$ for $\gamma\neq\beta_n+j$ with non-negative integers n and j. If we introduce such terms, then the following equality should hold:

$$0=4d[m^2-1+\pi\gamma\cot(\pi\gamma)](1-4x^2)^{\gamma-1}. \quad (4.5)$$

This demands that $d\equiv0$. In the following subsections, we will examine our new basis function in detail both analytically and numerically.

A. An analytic approach

In this subsection, we will restrict ourselves to the case where $N=1$. Up to $O(\epsilon^{\beta_1-1})$, we have four equations:

$$\epsilon^{\beta_0-1}:0=4(m^2-1)c_0^0+4\pi\beta_0\cot(\pi\beta_0)c_0^0, \quad (4.6a)$$

$$\epsilon^0:0=c_0^0\frac{\Gamma(1+\beta_0)}{(1-\beta_0)\Gamma(1/2+\beta_0)}-c_0^1\frac{\Gamma(2+\beta_0)}{\beta_0\Gamma(3/2+\beta_0)}$$
$$+c_1^0\frac{\Gamma(1+\beta_1)}{(1-\beta_1)\Gamma(1/2+\beta_1)}+\frac{\alpha}{4}\left[c_0^0\frac{\Gamma(1+\beta_0)}{\Gamma(3/2+\beta_0)}\right.$$
$$\left.+c_0^1\frac{\Gamma(2+\beta_0)}{\Gamma\left(\frac{5}{2}+\beta_0\right)}+c_1^0\frac{(1+\beta_1)}{\Gamma(3/2+\beta_1)}\right], \quad (4.6b)$$

$$\epsilon^{\beta_0}:c_0^0M^2=4c_0^1(m^2-1)+4\pi\cot(\pi\beta_0)$$
$$\times\left[-\frac{c_0^0\beta_0}{2}+(1+\beta_0)c_0^1\right], \quad (4.6c)$$

$$\epsilon^{\beta_1-1}:0=4(m^2-1)c_1^0+4\pi\beta_1\cot(\pi\beta_1)c_1^0. \quad (4.6d)$$

Equations (4.6a) and (4.6d) are automatically satisfied. Since $c_0^0\equiv1$, we can solve Eqs. (4.6b) and (4.6c) for c_0^1 and c_1^0 in terms of m, M, β_0, and β_1.

Now, in order to solve the above equations for M, we assume that all physical quantities can be expanded in terms of quark mass m. That is,

$$M^2=b_0+b_1m+\cdots. \quad (4.7)$$

Thus, we have, up to $O(m)$,

$$\int_{-1/2}^{1/2}dx|\Phi(x)|^2\equiv\langle\Phi|\Phi\rangle$$
$$=1+m[2\beta_0^1(22+8\beta_1^0[1-3\log(2)]$$
$$-12\log(2)+\alpha\{2-6\log(2)$$
$$-\beta_1^0[5-6\log(2)]\})]$$
$$\times\{3[-2+\alpha(-1+\beta_1^0)-4\beta_1^0]\}^{-1}$$
$$+b_1m\left[\frac{2(1-\beta_1^0)}{2+\alpha+4\beta_1^0-\alpha\beta_1^0}\right] \quad (4.8)$$

and

$$\int_{-1/2}^{1/2}\int_{-1/2}^{1/2}dx\,dy\,\Phi(x)H(x,y)\Phi(y)$$
$$\equiv\langle\Phi|H|\Phi\rangle$$
$$=\alpha+m(-2(1+2\beta_1^0)(3+\beta_0^{1^2}\pi^2)$$
$$+\alpha\{3(-1+\beta_1^0)+(\beta_0^1)^2[48+6\beta_1^0-\pi^2+\beta_1^0\pi^2$$
$$-36\log(2)-36\beta_1^0\log(2)]\})$$
$$\times\{3\beta_0^1[-2+\alpha(-1+\beta_1^0)-4\beta_1^0]\}^{-1}$$
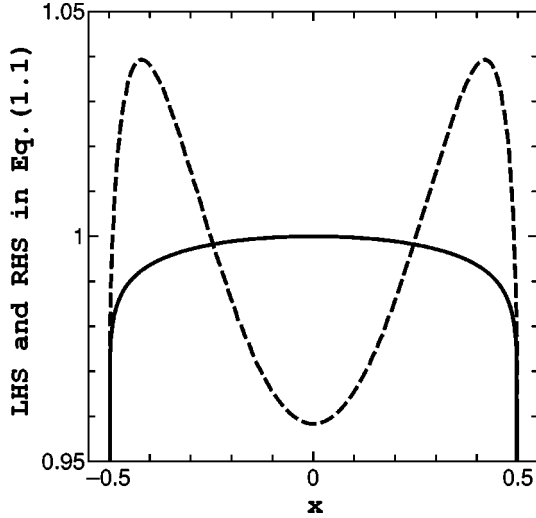$$+b_1m\left[\frac{2(1-\beta_1^0)}{2+\alpha+4\beta_1^0-\alpha\beta_1^0}\right]. \quad (4.9)$$

FIG. 2. The comparison of the relative values of both sides of Eq. (1.5) for $\alpha=0$. The wave function in Eq. (1.5) was approximated by Eq. (4.3) with $N=1$ and Eq. (4.10) with $m=0.01$. The solid line represents the LHS and the dashed line stands for the RHS.

We may expect that coefficient b_1 depends on α, as both coefficients of m in Eqs. (4.8) and (4.9) explicitly depend on α. However, this is not the case. Indeed, from Eq. (4.4), we are led to

$$M^2=\alpha+\left(\frac{1}{\beta_0^1}+\frac{\beta_0^1\pi^2}{3}\right)m+O(m^2)$$

$$=\alpha+\frac{2\pi}{\sqrt{3}}m+O(m^2). \tag{4.10}$$

Note that we obtained the first line in the above expansion without referring to the explicit value of β_1. We did not reproduce the result of Banks *et al.*, but that of Bergknoff. The approximated wave function in case $m=0.01$ and $N=1$ is shown in Fig. 2. The approximation used here is so rough that the coincidence of the LHS and the RHS is poor. Nevertheless, the behavior of the RHS near the end points is quite calm compared with the results of the conventional basis function method. The smoothness of the RHS is quite natural. As mentioned previously, the wave function given in Eq. (4.3) is the most general. If we truncate the wave function up to order N, it becomes smooth. So we may also expect the RHS of the 't Hooft–Bergknoff-Eller equation to be smooth.

We may expect that the coincidence of the LHS and the RHS will be improved if the higher order terms are included.

In cases where $N\geqslant2$, we cannot treat things analytically. We will attempt to solve Eq. (1.5) numerically by the use of new basis function in the next subsection.

B. A numerical approach

In general, we have

$$0=M^2\Phi(x)-\int_{-1/2}^{1/2}dy\,H(x,y)\Phi(y)\bigg|_{1-4x^2=\epsilon}$$

$$=-4\sum_{n=0}^{\infty}c_n^0(m^2-1+\pi\beta_n\cot\pi\beta_n)\epsilon^{\beta_n-1}$$

$$+\sum_{n=0}^{\infty}\sum_{j=0}^{\infty}\Bigg(M^2c_n^j-4(m^2-1)c_n^{j+1}$$

$$-4\pi\sum_{k=0}^{j+1}c_n^k(\beta_n+k)\cot\pi\beta_n\frac{(-1/2)_{j+1-k}}{(j+1-k)!}\Bigg)\epsilon^{\beta_n+j}$$

$$+\frac{\pi^{1/2}}{2}\sum_{n=0}^{\infty}\sum_{j=0}^{\infty}c_n^j\Bigg(\frac{4\Gamma(\beta_n+j+1)}{(\beta_n+j-1)\Gamma(\beta_n+j+1/2)}$$

$$-\frac{\alpha\Gamma(\beta_n+j+1)}{\Gamma(\beta_n+j+3/2)}\Bigg)\epsilon^0$$

$$+2\pi^{1/2}\sum_{k=0}^{\infty}\Bigg(\sum_{n=0}^{\infty}\sum_{j=0}^{\infty}c_n^j\frac{\Gamma(\beta_n+j+1)}{(\beta_n+j-1)\Gamma(\beta_n+j+1/2)}$$

$$\times\frac{(1/2-\beta_n-j)_{k+1}}{(2-\beta_n-j)_{k+1}}\Bigg)\epsilon^{k+1}. \tag{4.11}$$

Here, $\epsilon\equiv1-4x^2$ as before. Of course, the first line in Eq. (4.11) cancels automatically because of the definition of β_n's. Then, suppose that we truncate series in Eq. (4.3) to $O(\epsilon^{\beta_N})$. That is, we set $c_n^j=0$ for $n+j>N$. For a given m, we put $M^2=M_i^2$. We can then solve Eq. (4.11) for c_n^j in terms of M_i. We thus obtain the M_i dependent truncated wave function, say, $\Phi(x;M_i)$. We can calculate a new mass eigenvalue M_{i+1} using this wave function as

$$M_{i+1}^2=\frac{\langle\Phi(M_i)|H|\Phi(M_i)\rangle}{\langle\Phi(M_i)|\Phi(M_i)\rangle}. \tag{4.12}$$

We can use Eq. (4.10) as M_0^2. For $N\leqslant15$ and $m=0.01$, mass M^2 converges in 5 iterations. For $0<m<0.5$, we obtain M^2's which are summarized in Table II. We can fit them by polynomials, as follows:

TABLE II. Numerical results for bound state mass M^2 in SU(N_C) and U(N_C) models as a function of quark mass m.

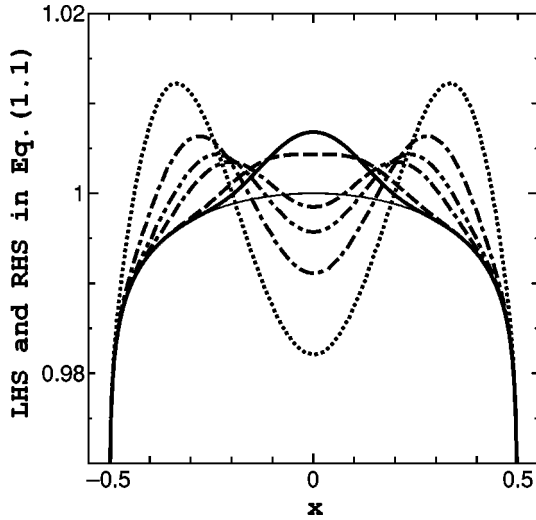| m	0.01	0.10	0.20	0.30	0.40	0.50
M^2 in SU(N_C)	0.036634	0.398634	0.869282	1.412358	2.028271	2.717415
M^2 in U(N_C)	1.036607	1.396177	1.860377	2.394130	2.998685	3.675073

FIG. 3. The convergence of the new basis function expansion for $m=0.01$ fixed. The thin solid line represents the LHS in Eq. (1.5), provided that the wave function was approximated by Eq. (4.3) with $N=15$. The dotted line denotes the RHS with $N=2$, the dot-dashed line exhibits the RHS with $N=3$, the dot-dot-dashed line represents the RHS with $N=4$, the dot-dash-dashed line stands for the RHS with $N=5$, and the dashed line exhibits the RHS with $N=10$. The thick solid line indicates the RHS in Eq. (1.5) with wave function given in Eq. (4.3) with $N=15$.

$$M^2(\alpha=0,m)=3.62763m+3.58027m^2+0.0683573m^3$$
$$+O(m^4)$$

$$M^2(\alpha=1,m)=1+3.62421m+3.34492m^2+0.213839m^3$$
$$+O(m^4). \qquad (4.13)$$

It should be noted here that the coefficients of m are consistent with Eq. (4.10) and Bergknoff's result. In order to see the efficacy of this new basis function expansion, we show the wave functions in Fig. 3.

V. SUMMARY AND DISCUSSION

In the preceding sections we have introduced the new basis function and calculated the mass eigenvalue of the bound state using the new basis function. We have found that (1) the new basis function gives an effective approximation of the wave function and (2) the mass eigenvalues are consistent with the results of the precursors. In the remainder of this section, we will discuss the 2% discrepancy problem.

Let us consider the wave function given by

$$\Phi(x)=(1-4x^2)^{\gamma_0}+a_1(1-4x^2)^{\gamma_1}, \qquad (5.1)$$

where we assume only $\gamma_0=\gamma_0^1 m+\gamma_1^3 m^3+O(m^5)$, $\gamma_1=\gamma_1^0+\gamma_1^1 m+O(m^2)$, and $a_1=a_1^1 m^{1/2+\delta}+O(m)$. We are then led to

$$M^2\equiv\langle\Phi|H|\Phi\rangle/\langle\Phi|\Phi\rangle=\alpha+\left(\frac{1}{\gamma_0^1}+\frac{\gamma_0^1\pi^2}{3}\right)m+O(m^{1+2\delta}).$$
$$(5.2)$$

This relation holds independently of the details of γ_0, γ_1, and a_1, except for certain assumptions which were made before Eq. (5.2). The coefficient of m in Eq. (5.2) has the minimum value $2\pi/\sqrt{3}$ when $\gamma_0^1=\beta_0^1\equiv\sqrt{3}/\pi$. We may therefore conclude that Eq. (5.2) holds universally, provided that (1) the wave function can be expanded as a power series of $(1-4x^2)$, similar to

$$\Phi(x)=(1-4x^2)^{\gamma_0}+\sum_{j=1}^{\infty}a_j(1-4x^2)^{\gamma_j}, \quad \gamma_0<\gamma_1<\cdots$$

and (2) the coefficients of the series, a_j's, are of order $m^{1/2+\delta}$.

An almost identical result has been obtained by Harada et al. [15], in which they have restricted themselves to a case where $\gamma_0=\beta_0$, and $\gamma_j=\beta_0+j$. Our conclusion is a generalization of Harada et al.'s result. We, therefore, cannot solve the "2% discrepancy" problem in the context of the 't Hooft-Bergknoff-Eller equations.

APPENDIX: PROOF OF EQ. (3.2)

We will prove Eq. (3.2). For monomial x^n, by the use of the definition of the finite part integral, we obtain

$$\mathcal{L}:x^n\mapsto-\frac{4x^n}{1-4x^2}+nx^{n-1}\log\frac{1-2x}{1+2x}+f_n(x). \quad (A1)$$

Here,

$$f_m(x)=\sum_{k=2}^{m}\frac{{}_mC_k}{k-1}\left\{\left(\frac{1}{2}-x\right)^{k-1}-\left(-\frac{1}{2}-x\right)^{k-1}\right\}x^{m-k}$$

$$=\begin{cases}\displaystyle\sum_{k=0}^{n-1}\frac{(2k+1)x^{2k}}{\{2(n-k)-1\}2^{2n-2k-2}}, & \text{for } m=2n,\\[3mm]\displaystyle\sum_{k=0}^{n-1}\frac{(2k+2)x^{2k+1}}{\{2(n-k)-1\}2^{2n-2k-2}}, & \text{for } m=2n+1.\end{cases}$$
$$(A2)$$

Using identities

$$\sum_{n=0}^{\infty}c_{2n}\left(-\frac{4x^{2n}}{1-4x^2}\right)=-\sum_{n=0}^{\infty}\sum_{k=0}^{n}c_{2k}4^{n-k+1}x^{2n},$$
$$(A3)$$

$$\sum_{n=0}^{\infty} 2n c_{2n} x^{2n-1} \log\frac{1-2x}{1+2x}$$

$$= -\sum_{n=0}^{\infty} \left[\sum_{k=0}^{n} c_{2k}\frac{2^{2n-2k+3}k}{2(n-k)+1}\right] x^{2n}, \quad \text{(A4)}$$

and

$$\sum_{n=1}^{\infty} c_{2n} f_{2n}(x) = -\sum_{n=0}^{\infty} \left[\sum_{k=0}^{n} c_{2k}\frac{(2n+1)2^{2n-2k+1}}{2(n-k)+1}\right] x^{2n}, \quad \text{(A5)}$$

we see that, for a given even function $\sum_{n=0}^{\infty} c_{2n} x^{2n}$,

$$\mathcal{L}: \sum_{n=0}^{\infty} c_{2n} x^{2n} \mapsto -\sum_{n=0}^{\infty} (2n+1)\sum_{k=0}^{\infty}\frac{4^{n-k+1} c_{2k}}{2(n-k)+1} x^{2n}. \quad \text{(A6)}$$

Power series expansion

$$(1-4x^2)^{\beta} = \sum_{n=0}^{\infty}\frac{(-\beta)_n(4x^2)^n}{n!}, \quad \text{(A7)}$$

holds where $(a)_n \equiv \Gamma(a+n)/\Gamma(a)$ is the Pochhammer symbol. Thus, for $c_{2n} = (-\beta)_n 4^n/n!$ and $c_{2n+1} = 0$, we see that

$$\mathcal{L}:(1-4x^2)^{\beta} \mapsto -\frac{4\pi^{1/2}\Gamma(1+\beta)}{\Gamma(1/2+\beta)}F(1,1/2-\beta;1/2;4x^2), \quad \text{(A8)}$$

where $F(a,b;c;z)$ is the Gauss' hypergeometric function. Analogously, we have

$$\mathcal{L}:x(1-4x^2)^{\beta} \mapsto -\frac{8\pi^{1/2}\Gamma(1+\beta)}{\Gamma(1/2+\beta)}xF(2,1/2-\beta;3/2;4x^2). \quad \text{(A9)}$$

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[1] S. J. Brodsky, H-C. Pauli, and S. Pinsky, Phys. Rep. **301**, 299 (1998).

[2] M. Burkardt, Adv. Nucl. Phys. **23**, 1 (1996).

[3] I. Tamm, J. Phys. (Moscow) **9**, 449 (1945); S. M. Dancoff, Phys. Rev. **78**, 382 (1950).

[4] R. J. Perry, A. Harindranath, and K. G. Wilson, Phys. Rev. Lett. **65**, 2959 (1990).

[5] Y. Ma and J. R. Hiller, J. Comput. Phys. **82**, 229 (1989).

[6] Y. Mo and R. J. Perry, J. Comput. Phys. **108**, 159 (1993).

[7] K. Harada, T. Sugihara, M. Taniguchi, and M. Yahiro, Phys. Rev. D **49**, 4226 (1994).

[8] S. Coleman, R. Jackiw, and L. Susskind, Ann. Phys. (N.Y.) **93**, 267 (1975); S. Coleman, *ibid.* **101**, 239 (1976).

[9] J. Schwinger, Phys. Rev. **128**, 2425 (1962).

[10] H. Bergknoff, Nucl. Phys. **B122**, 215 (1977).

[11] T. Sugihara, M. Matsuzaki, and M. Yahiro, Phys. Rev. D **50**, 5274 (1994).

[12] G. 't Hooft, Nucl. Phys. **B75**, 461 (1974).

[13] K. Hornbostel, S. J. Brodsky, and H.-C. Pauli, Phys. Rev. D **41**, 3814 (1990).

[14] T. Banks, J. Kogut, and L. Susskind, Phys. Rev. D **13**, 1043 (1976).

[15] K. Harada, T. Heinzl, and C. Stern, Phys. Rev. D **57**, 2460 (1998).

[16] S. Huang, J. W. Negele, and J. Polonyi, Nucl. Phys. **B307**, 669 (1988).

[17] B. van de Sande, Phys. Rev. D **54**, 6347 (1996).

[18] A. Duncan, S. Pernice, and E. Schnapka, Phys. Rev. D **55**, 2422 (1997).

[19] O. Abe, G. J. Aubrecht, and K. Tanaka, Phys. Rev. D **56**, 2242 (1997).

[20] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables* (Dover, New York, 1972).